

# Integrable matrix equations related to pairs of compatible associative algebras

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## Abstract

We study associative multiplications in semi-simple associative algebras over  $\mathbb{C}$  compatible with the usual one. An interesting class of such multiplications is related to the affine Dynkin diagrams of  $\tilde{A}_{2k-1}$ ,  $\tilde{D}_k$ ,  $\tilde{E}_6$ ,  $\tilde{E}_7$ , and  $\tilde{E}_8$ -type. In this paper we investigate in details the multiplications of the  $\tilde{A}_{2k-1}$ -type and integrable matrix ODEs and PDEs generated by them.

MSC numbers: 17B80, 17B63, 32L81, 14H70

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# Introduction

In the papers [1, 2, 3, 4] different applications of the notion of compatible Lie brackets to the integrability theory have been considered. Various odd examples of compatible Lie structures have been presented in [5, 1, 2].

A pair of compatible associative multiplications is more rigid algebraic structure than a pair of compatible Lie brackets and therefore there is a chance to construct more developed theory and some series of important examples of such multiplications.

In [6] we have studied multiplications compatible with the matrix multiplication or, in other words, linear deformations of the matrix product. It turns out that these deformations of the matrix algebra are in one-to-one correspondence with representations of certain algebraic structures, which we call  $M$ -structures. The case of direct sum of several matrix algebras corresponds to representations of the so-called  $PM$ -structures (see [6]).

The main result of [6] is a description of an important class of  $M$  and  $PM$ -structures. The classification of these structures naturally leads to the Cartan matrices of affine Dynkin diagrams of the  $\tilde{A}_{2k-1}$ ,  $\tilde{D}_k$ ,  $\tilde{E}_6$ ,  $\tilde{E}_7$ , and  $\tilde{E}_8$ -type.

In this paper, we investigate integrable equations related to the affine Dynkin diagrams of  $\tilde{A}$ -type in details. Recall some results of the papers [4, 6, 1] adopted for our goals. For simplicity we formulate these results for the matrix algebra  $Mat_n$  but all of them remain to be true for any semi-simple associative algebra.

A multiplication  $\circ$  defined on the vector space  $Mat_n$  of all  $n \times n$  matrices is said to be compatible with the matrix multiplication if the product

$$X \bullet Y = XY + \lambda X \circ Y \quad (0.1)$$

is associative for any constant  $\lambda$ .

Since the matrix algebra is rigid, the multiplication (0.1) is isomorphic to the matrix multiplication for almost all values of the parameter  $\lambda$ . This means that there exists a formal series of the form

$$S_\lambda = \mathbf{1} + R \lambda + O(\lambda^2), \quad (0.2)$$

with the coefficients being linear operators on  $Mat_n$ , such that

$$S_\lambda(X) S_\lambda(Y) = S_\lambda(XY + \lambda X \circ Y). \quad (0.3)$$

It follows from this formula that the multiplication  $\circ$  is given by

$$X \circ Y = R(X)Y + XR(Y) - R(XY). \quad (0.4)$$

The series  $S_\lambda$  is defined up to a transformation  $S_\lambda \rightarrow T_\lambda S_\lambda T_\lambda^{-1}$ , where  $T_\lambda$  is an arbitrary element of  $Mat_n$  of the form  $T_\lambda = \mathbf{1} + t\lambda + O(\lambda^2)$ . This leads to the transformation

$$R \rightarrow R + ad_t, \quad (0.5)$$

which does not change the multiplication (0.4).

In the matrix case it is convenient to write down the operator  $R$  in the form

$$R(x) = a_1 x b^1 + \dots + a_p x b^p + c x, \quad a_i, b^i, c \in Mat_n \quad (0.6)$$

with  $p$  being smallest possible.

A general construction from [4] establishes a relationship between pairs of compatible associative multiplications and integrable top-like systems. Let  $\circ$  be a multiplication (0.4) compatible with the matrix product. Consider the following matrix differential equation

$$\frac{dx}{dt} = [R(x) + R^*(x), x], \quad (0.7)$$

where  $*$  stands for the adjoint operator with respect to the bi-linear form trace  $(xy)$ . This means that, if  $Q$  is a linear operator in  $Mat_n$ , then  $Q^*$  is defined by the equation  $\text{trace}(Q(x)y) = \text{trace}(xQ^*(y))$  for all  $x, y \in Mat_n$ . In particular,

$$R^*(x) = b^1 x a_1 + \dots + b^p x a_p + x c.$$

It turns out [4] that equation (0.7) possesses the Lax representation

$$\frac{dL}{dt} = [A, L],$$

where

$$L = (S_\lambda^{-1})^* (x), \quad A = \frac{1}{\lambda} S_\lambda(x). \quad (0.8)$$

To make these formulas constructive, we should find  $S_\lambda$  and  $S_\lambda^{-1}$  in a closed form i.e. as analytic operator-valued functions. As usual, the integrals of motion for (0.7) are given by coefficients of different powers of  $\lambda$  in  $\text{trace}(L^k)$ ,  $k = 1, 2, \dots$

The simplest example of a multiplication compatible with the matrix product corresponds to the Dynkin diagram  $\tilde{A}_1$ . This multiplication  $\circ$  is given by  $x \circ y = xcy$ , where  $c$  is an arbitrary matrix. In this case we have

$$R(x) = cx, \quad S_\lambda(x) = (1 + \lambda c) x, \quad L = x(1 + \lambda c)^{-1}.$$

The Lax equation is equivalent to the well known integrable matrix equation

$$\frac{dx}{dt} = x^2 c - c x^2. \quad (0.9)$$

The simplest integrals for (0.9) are given by  $H_{k,0} = \text{trace}(x^k)$ . Moreover, the  $L$ -operator produces an infinite set of homogeneous integrals  $H_{i,j}$ , where  $i$  and  $j$  are degrees with respect to  $x$  and  $c$ . For example,

$$H_{1,1} = \text{trace}(xc), \quad H_{2,1} = \text{trace}(x^2 c), \quad H_{2,2} = \text{trace}(2c^2 x^2 + cxcx). \quad (0.10)$$

Equation (0.9) is Hamiltonian one with respect to the standard matrix linear Poisson bracket given by the Hamiltonian operator  $ad_x$  and Hamiltonian function  $H_{2,1}$ . Indeed, (0.9) can be rewritten in the form (0.7) as  $x_t = [x, xc + cx]$ .

Matrix equations of arbitrary size like (0.9) are important because of possibility to make different reductions. For the most trivial reduction one may regard  $x$  as a block-matrix. In this case (0.9) becomes a system of several matrix equations for the block entries of  $x$ . Under reduction  $x^T = -x$ ,  $c^T = c$  (0.9) is a commuting flow for the  $n$ -dimensional Euler equation [7, 8, 9]. Another reduction of (0.9) was mentioned in [9]. Let  $x$  and  $c$  in (0.9) be represented by matrices of the form

$$x = \begin{pmatrix} 0 & u_1 & 0 & 0 & \cdot & 0 \\ 0 & 0 & u_2 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & u_{N-1} \\ u_N & 0 & 0 & 0 & \cdot & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 0 & 0 & \cdot & 0 & J_N \\ J_1 & 0 & 0 & \cdot & 0 & 0 \\ 0 & J_2 & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & J_{N-1} & 0 \end{pmatrix}, \quad (0.11)$$

where  $u_k$  and  $J_k$  are block matrices (of any dimension). It follows from equation (0.9) that  $u_k$  satisfy the non-abelian Volterra equation

$$\frac{d}{dt}u_k = u_k \circ u_{k+1} \circ J_{k+1} - J_{k-1} \circ u_{k-1} \circ u_k, \quad k \in \mathbb{Z}_N.$$

Integrable equation (0.9) contains one arbitrary constant matrix  $c$ . Such equations have been systematically investigated in [9]. In Sections 1,2 we present a series of integrable matrix equations, which depend on two constant matrices related by certain algebraic relations providing the integrability. In Section 3 these results are generalized to the case of several matrix unknowns. Skew-symmetric reductions as well as reductions of (0.11)-type are available for our matrix ODEs.

Different applications of compatible associative multiplications are related to integrable deformations of the principle  $GL(n)$ -chiral model (see [1]). Let  $\circ$  be associative multiplication compatible with the  $Mat_n \oplus Mat_n$  product,  $S_\lambda$  be the series (0.2) with the property (0.3). Define operators  $T_1, T_2$  on  $Mat_n$  by means of the following decompositions

$$S_\lambda(u, 0) = (u, 0) + \lambda(\cdot, T_1(u)) + O(\lambda^2), \quad S_\lambda(0, v) = (0, v) + \lambda(T_2(v), \cdot) + O(\lambda^2). \quad (0.12)$$

Then the following hyperbolic system

$$u_t = [u, T_2(v)], \quad v_\tau = [v, T_1(u)], \quad u, v \in Mat_n \quad (0.13)$$

possesses a zero-curvature representation  $[L, M] = 0$ , where

$$L = \frac{d}{dy} + \frac{1}{\lambda}S_\lambda(u, 0), \quad M = \frac{d}{dx} + \frac{1}{\lambda}S_\lambda(0, v).$$

If  $T_1 = T_2 = \mathbf{1}$ , then (0.13) is just the principle chiral model [10]. In Section 3 we describe a class of integrable models (0.13) related to  $PM$ -structures of the  $\tilde{A}_{2k-1}$ -type.

# 1 Associative product of the $\tilde{A}_3$ -type.

The  $M$ -structure related to  $\tilde{A}_3$  (see [6] ) is defined by two arbitrary constant matrices  $A$  and  $B$  such that

$$A^2 = B^2 = \mathbf{1}. \quad (1.14)$$

The corresponding associative multiplication is given by (0.4), where

$$R(x) = AxB + BAx. \quad (1.15)$$

This structure leads to the following integrable matrix equation

$$x_t = [x, BxA + AxB + xBA + BAx]. \quad (1.16)$$

The Lax representation (0.8) for (1.16) is given by the following explicit formulas for  $S_\lambda$  and  $S_\lambda^{-1}$ :

$$\begin{aligned} S_\lambda(x) &= \frac{1-q}{2} BxB + \frac{1+q}{2} x + \lambda(AxB + BAx), \\ S_\lambda^{-1}(x) &= \frac{1}{q}(\mathbf{1} + \lambda K)^{-1} \left( \frac{q-1}{2} BxB + \frac{1+q}{2} x + \lambda(ABx - AxB) \right), \end{aligned}$$

where  $q = \sqrt{1 - 4\lambda^2}$ ,  $K = AB + BA$ . Note that both  $A$  and  $B$  commute with  $K$ .

The canonical form (1.15) for the operator  $R$  with respect to transformations (0.5) is homogeneous in  $A$  and  $B$  that gives rise to short expressions for  $S_\lambda$  and  $S_\lambda^{-1}$  irrationally depending on  $\lambda$ . A different form

$$\bar{R}(x) = AxB + BAx + Bx - xB \quad (1.17)$$

provides the simplest form of the operator (0.2) with the property (0.3):

$$\bar{S}_\lambda(x) = x + \lambda \bar{R}(x) \quad (1.18)$$

but the expression for  $\bar{S}_\lambda^{-1}$  is more complicated:

$$\begin{aligned} \bar{S}_\lambda^{-1}(x) &= \frac{1}{4\lambda^2 - 1} (\mathbf{1} + \lambda K)^{-1} \left( \lambda (AxB + Bx + BAx) + 2\lambda^2 (BxB + BAxB + Ax) \right. \\ &\quad \left. - (\lambda + 2\lambda^2 K) xB - (1 + \lambda K - 2\lambda^2) x \right). \end{aligned}$$

The corresponding Lax operators (0.8) are rational in  $\lambda$ .

The simplest linear and quadratic first integrals for (1.16) generated by the  $L$ -operator are given by

$$H_{1,1} = \text{trace}[x(AB+BA)], \quad H_{1,2} = \text{trace}[x(ABAB+BABA)], \quad H_{2,1} = \text{trace}[BAx^2 + AxBx],$$

and

$$H_{2,2} = \text{trace}[2BABAx^2 + 2ABABx + 2BABx + ABxABx + BAxBx].$$

**Remark.** Equation (1.16) is the simplest example of matrix integrable equation with constant matrices related by constraints. If we assume that  $A, B, x$  are block-matrices of the form:

$$A = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad B = \begin{pmatrix} P & \mathbf{1} + P \\ \mathbf{1} - P & -P \end{pmatrix}, \quad x = \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix},$$

then we obtain an integrable ODE with one arbitrary constant matrix  $P$  and four matrix unknowns  $x_{i,j}$ . Furthermore, assuming that  $P$  is scalar matrix, we obtain an integrable matrix ODE with constant coefficients.

Equation (1.16) admits the following skew-symmetric reduction

$$x^T = -x, \quad B = A^T. \quad (1.19)$$

Different integrable  $so(n)$ -models provided by reduction (1.19) are in one-to-one correspondence with equivalence classes with respect to the  $SO(n)$  gauge action of  $n \times n$  matrices  $A$  such that  $A^2 = \mathbf{1}$ . For the real matrix  $A$ , a canonical form for such equivalence class can be chosen as

$$A = \begin{pmatrix} \mathbf{1}_p & T \\ 0 & \mathbf{1}_{n-p} \end{pmatrix}. \quad (1.20)$$

Here  $\mathbf{1}_s$  stands for the unity  $s \times s$  matrix and  $T = (t_{ij})$ , where  $t_{ij} = \delta_{ij}\alpha_i$ . This canonical form is defined by the discrete natural parameter  $p$  and continuous parameters  $\alpha_1, \dots, \alpha_r$ , where  $p \leq n/2$ ,  $r = \min(p, n-p)$ . A different canonical form for  $A$  will be given in the next section.

For example, in the case  $n = 4$  the equivalence classes with  $p = 2$  and  $p = 1$  give rise to the Steklov and the Poincare integrable models [11, 12], correspondingly. Thus, whereas (0.9) is a matrix version of the  $so(4)$  Schottky-Manakov top [13, 7], equation (1.16)-(1.20) with  $p = [n/2]$  and  $p = 1$  can be regarded as  $so(n)$  generalizations for the  $so(4)$  Steklov and Poincare models (for different generalization see [14, 15]).

## 2 Associative products of the $\tilde{A}_{2k-1}$ -type. Case of one matrix variable.

Let  $A, B, C \in Mat_n$  be matrices that satisfy the following set of relations

$$A^k = B^k = \mathbf{1}, \quad (2.21)$$

$$B^i A^j = \frac{\varepsilon^{-j} - 1}{\varepsilon^{-i-j} - 1} A^{i+j} + \frac{\varepsilon^i - 1}{\varepsilon^{i+j} - 1} B^{i+j}, \quad i + j \not\equiv 0 \pmod{k}, \quad (2.22)$$

$$B^i A^{k-i} = \mathbf{1} + (\varepsilon^i - 1)C, \quad (2.23)$$

where  $\varepsilon = \exp(2\pi i/k)$ . It follows from results of [6] that formula (0.4) with

$$R(x) = \sum_{i=1}^{k-1} \frac{1}{\varepsilon^i - 1} A^{k-i} x B^i + Cx,$$

defines the associative multiplication  $\circ$  compatible with the standard matrix product in  $Mat_n$ . According to [6] this multiplication corresponds to the affine Dynkin diagram of the  $\tilde{A}_{2k-1}$ -type.

Let  $T$  be any matrix such that  $T^k - \mathbf{1}$  is invertible and

$$AT = \varepsilon T A. \quad (2.24)$$

Then

$$B = (\varepsilon T - \mathbf{1})(T - \mathbf{1})^{-1} A, \quad C = T(T - \mathbf{1})^{-1} \quad (2.25)$$

satisfy identities (2.21)-(2.23). This is a convenient way to resolve this system of identities.

**Proposition 1.** The operator  $S_\lambda$  with the property (0.3) is defined by

$$S_\lambda(x) = \sum_{i=0}^{k-1} P_i x B^i, \quad P_i = \frac{\lambda}{\varepsilon^i \mu - 1} (\mu T - \mathbf{1})(T - \mathbf{1})^{-1} A^{k-i}, \quad (2.26)$$

where

$$\mu = \left( \frac{2 + (k+1)\lambda}{2 - (k-1)\lambda} \right)^{1/k}.$$

The inverse operator is given by the formula

$$S_\lambda^{-1}(x) = \sum_{i=0}^{k-1} Q_i x B^i, \quad Q_i = \frac{(\mu^k - 1)^2}{\lambda k^2 \mu^{k-1} (\mu - \varepsilon^i)} (\varepsilon^{k-i} T - \mathbf{1})(\varepsilon^{k-i} \mu T - \mathbf{1})^{-1} A^{k-i}. \quad (2.27)$$

The corresponding Lax operator (0.8) is given by

$$L(x) = \sum_{i=0}^{k-1} B^i x Q_i. \quad (2.28)$$

**Remark.** If we take operator  $R$  in the following equivalent but different form

$$\bar{R}(x) = \sum_{i=1}^{k-1} \frac{1}{\varepsilon^i - 1} (A^{k-i} - 1) x B^i + (C + \sum_{i=1}^{k-1} \frac{1}{\varepsilon^i - 1} B^i) x,$$

then the operator  $S_\lambda$  has the simplest form (1.18), but the formula for  $S_\lambda^{-1}(x)$  is more complicated.

Using Proposition 1, one can calculate the integrals of motion for equation (0.7) as coefficients of trace  $(L^k)$ ,  $k = 1, 2, \dots$ . For example, the simplest quadratic integral is given by

$$H = \text{trace}(xR(x)).$$

Equation (0.7) can be written in the Hamiltonian form as

$$\frac{dx}{dt} = [x, \text{grad } H].$$

The skew-symmetric reduction is given by

$$B = (A^t)^{-1}, \quad C = \frac{\varepsilon}{1 - \varepsilon} (A^t A - 1),$$

where  $A$  should satisfy the following additional constraint

$$(A^t A - \varepsilon^{-1}) A (A^t A - 1) = \varepsilon (A^t A - 1) A (A^t A - \varepsilon^{-1}).$$

It turns out that

$$A = \sqrt{z_1} e_{k,1} + \sum_{i=2}^k \sqrt{z_i} e_{i-1,i}, \quad (2.29)$$

where  $z_{j+1} = f(z_j)$ ,

$$f(z) = \frac{1}{1 + \varepsilon - \varepsilon z},$$

defines a skew-symmetric reduction with one arbitrary parameter  $z_1$ . In particular, for  $k = 2$  we have

$$A = \begin{pmatrix} 0 & \alpha \\ \alpha^{-1} & 0 \end{pmatrix},$$

where  $\alpha$  is an arbitrary parameter.

**Remark.** Any block-diagonal matrix with blocks (of different size) of the form (2.29) also provides a non-trivial skew-symmetric reduction. If we take  $p$  blocks of the size 1 and the remaining blocks of the size 2, we get a canonical form for matrix  $A$  equivalent but different from (1.20).

### 3 Associative products of the $\tilde{A}_{2k-1}$ -type. Case of $m$ matrix variables.

In this section we consider associative multiplications compatible with the product in the direct sum of  $m$  copies of  $Mat_n$ . We use the notation  $x = (x_1, \dots, x_m)$ , where  $x_1, \dots, x_m \in Mat_n$ , for  $x \in (Mat_n)^m$ . The standard multiplication in  $(Mat_n)^m$  is given by the formula  $xy = (x_1 y_1, \dots, x_m y_m)$ . Using results obtained in [6], the following statement can be proved.

**Proposition 2.** Let  $T, B$  be matrices such that

$$B^k = \mathbf{1}, \quad BT = \varepsilon TB.$$



Fix generic numbers  $\lambda_1, \dots, \lambda_m, t_1, \dots, t_m \in \mathbb{C}$  such that  $T^k - \lambda_\alpha^k$  are invertible for all  $\alpha = 1, \dots, m$ . Then formula (0.4) with

$$R(x) = (R_1(x), \dots, R_m(x)),$$

$$R_\alpha(x) = \sum_{(i,\beta) \neq (0,\alpha)} \frac{t_\beta}{\varepsilon^i \lambda_\beta - \lambda_\alpha} \cdot \frac{T - \lambda_\alpha}{\varepsilon^{-i} T - \lambda_\beta} B^{-i} x_\beta B^i + \left( \frac{t_\alpha}{T - \lambda_\alpha} - \sum_{(i,\beta) \neq (0,\alpha)} \frac{\varepsilon^i t_\beta}{\varepsilon^i \lambda_\beta - \lambda_\alpha} \right) x_\alpha$$

defines the following associative product compatible with the standard product in  $(Mat_n)^m$  :

$$x \circ y = ((x \circ y)_1, \dots, (x \circ y)_m),$$

where

$$(x \circ y)_\alpha = \sum_{(i,\beta) \neq (0,\alpha)} \frac{t_\beta}{\varepsilon^i \lambda_\beta - \lambda_\alpha} \left( \frac{T - \lambda_\alpha}{\varepsilon^{-i} T - \lambda_\beta} B^{-i} x_\beta B^i y_\alpha + x_\alpha \frac{T - \lambda_\alpha}{\varepsilon^{-i} T - \lambda_\beta} B^{-i} y_\beta B^i - \frac{T - \lambda_\alpha}{\varepsilon^{-i} T - \lambda_\beta} B^{-i} x_\beta y_\beta B^i \right) + t_\alpha x_\alpha \frac{1}{T - \lambda_\alpha} y_\alpha - \sum_{(i,\beta) \neq (0,\alpha)} \frac{\varepsilon^i t_\beta}{\varepsilon^i \lambda_\beta - \lambda_\alpha} x_\alpha y_\alpha.$$

**Remark.** The formulas from Section 2 after elimination of matrices  $A^i$ ,  $C$  via (2.25) coincides with the corresponding formulas from Proposition 2 with  $m = 1$ .

**Proposition 3.** For the multiplication from Proposition 2 the operator (0.2) with the property (0.3) is given by

$$S_\lambda(x) = (S_1(x), \dots, S_m(x)),$$

where

$$S_\alpha(x) = \sum_{i,\beta} \frac{\lambda t_\beta (T - \mu_\alpha)}{(\varepsilon^i \lambda_\beta - \mu_\alpha)(\varepsilon^{-i} T - \lambda_\beta)} B^{-i} x_\beta B^i.$$

The Lax operator (0.8) is defined by the formulas:

$$L(x) = (L_1(x), \dots, L_m(x)),$$

where

$$L_\alpha(x) = - \sum_{i,\beta} \frac{\prod_s (\lambda_\alpha^k - \mu_s^k)(\lambda_s^k - \mu_\beta^k)}{k^2 \lambda_\alpha^{k-1} \mu_\beta^{k-1} (\lambda_\alpha - \varepsilon^i \mu_\beta) \prod_{s \neq \alpha} (\lambda_\alpha^k - \lambda_s^k) \prod_{s \neq \beta} (\mu_\beta^k - \mu_s^k)} \cdot B^i x_\beta \frac{T - \lambda_\alpha}{t_\alpha \lambda (\varepsilon^{-i} T - \mu_\beta)} B^{-i}.$$

Here  $\mu_\alpha = \mu_\alpha(\lambda)$  are algebraic functions in  $\lambda$  such that  $\mu_\alpha(0) = \lambda_\alpha$  and

$$\sum_\gamma \frac{t_\gamma \lambda_\gamma^{k-1}}{\lambda_\gamma^k - \mu_\alpha^k} = \frac{1}{k\lambda}, \quad \alpha = 1, \dots, m.$$

**Remark.** In some sense, the Lax operator constructed in Proposition 3 is a multi-pole analog of operator (2.28).

To find the explicit form of integrable equations related to this Lax operator, one needs several first coefficients of the Taylor expansions of  $S_\lambda$  and  $L$ . We have

$$\begin{aligned}\mu_\alpha &= \lambda_\alpha - t_\alpha \lambda - t_\alpha \left( \frac{k-1}{2} \cdot \frac{t_\alpha}{\lambda_\alpha} + k \sum_{\gamma \neq \alpha} \frac{t_\gamma \lambda_\gamma^{k-1}}{\lambda_\gamma^k - \lambda_\alpha^k} \right) \lambda^2 + O(\lambda^3), \\ S_\alpha(x) &= x_\alpha + \left( \frac{t_\alpha}{T - \lambda_\alpha} - \frac{k-1}{2} \cdot \frac{t_\alpha}{\lambda_\alpha} - k \sum_{\gamma \neq \alpha} \frac{t_\gamma \lambda_\gamma^{k-1}}{\lambda_\gamma^k - \lambda_\alpha^k} \right) x_\alpha \lambda + \\ &\quad \sum_{(i,\beta) \neq (0,\alpha)} \frac{t_\beta (T - \lambda_\alpha)}{(\varepsilon^i \lambda_\beta - \lambda_\alpha)(\varepsilon^{-i} T - \lambda_\beta)} B^{-i} x_\beta B^i \lambda + O(\lambda^2), \\ L_\alpha(x) &= x_\alpha - x_\alpha \left( \frac{t_\alpha}{T - \lambda_\alpha} - \frac{k-1}{2} \cdot \frac{t_\alpha}{\lambda_\alpha} - k \sum_{\gamma \neq \alpha} \frac{t_\gamma \lambda_\gamma^{k-1}}{\lambda_\gamma^k - \lambda_\alpha^k} \right) \lambda \\ &\quad - \sum_{(i,\beta) \neq (0,\alpha)} B^i x_\beta \frac{t_\beta (T - \lambda_\alpha)}{(\varepsilon^i \lambda_\beta - \lambda_\alpha)(\varepsilon^{-i} T - \lambda_\beta)} B^{-i} \lambda + O(\lambda^2).\end{aligned}$$

The integrable system (0.7) has the form

$$\begin{aligned}\frac{dx_\alpha}{dt} &= \left[ \frac{t_\alpha}{T - \lambda_\alpha} x_\alpha + x_\alpha \frac{t_\alpha}{T - \lambda_\alpha} + \right. \\ &\quad \left. + \sum_{(i,\beta) \neq (0,\alpha)} \left( \frac{t_\beta (T - \lambda_\alpha)}{(\varepsilon^i \lambda_\beta - \lambda_\alpha)(\varepsilon^{-i} T - \lambda_\beta)} B^{-i} x_\beta B^i + B^i x_\beta \frac{t_\beta (T - \lambda_\alpha)}{(\varepsilon^i \lambda_\beta - \lambda_\alpha)(\varepsilon^{-i} T - \lambda_\beta)} B^{-i} \right), x_\alpha \right].\end{aligned}$$

It can be written in the Hamiltonian form as

$$\frac{dx}{dt} = [x, \text{grad } H],$$

where  $H = \sum_\alpha \text{trace}(x_\alpha R_\alpha(x))$ .

**Remark.** All these constructions including the integrable system are valid for the case  $k = 1$  also. In this case  $B = \mathbf{1}$ ,  $\varepsilon = 1$  and  $T$  is an arbitrary matrix. If also  $m = 1$ , then the system is equivalent to (0.9).

Another application of Propositions 2,3 in the case  $m = 2$  is the integrable PDE (0.13). To write down this system explicitly, we need to find the operators  $T_1$  and  $T_2$  defined by (0.12). The asymptotic formula for  $S_\lambda$  given above yields

$$T_1(x) = \sum_i \frac{t_1(T - \lambda_2)}{(\varepsilon^i \lambda_1 - \lambda_2)(\varepsilon^{-i} T - \lambda_1)} B^{-i} x B^i, \quad T_2(x) = \sum_i \frac{t_2(T - \lambda_1)}{(\varepsilon^i \lambda_2 - \lambda_1)(\varepsilon^{-i} T - \lambda_2)} B^{-i} x B^i.$$

## 4 Conclusion.

In this paper we have described representations of  $M$  and  $PM$ -structures related to the affine Dynkin diagrams of the  $\tilde{A}_{2k-1}$ -type and have presented Lax operators for the corresponding matrix top-like systems. Actually, these systems are bi-hamiltonian models [16]. We are planning to describe the Hamiltonian properties of these systems in a separate paper. Besides of that representations of  $PM$ -structures related to the affine Dynkin diagrams of the  $\tilde{D}_k, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ -types and the corresponding integrable models should be described explicitly. A systematic investigation of reductions of matrix models constructed in this paper seems to be an interesting problem. In this paper we have considered also integrable PDEs of the chiral model type related to  $PM$ -structures with  $m = 2$ . Compatible associative multiplications described by  $M$  and  $PM$ -structures can be used for construction of integrable PDEs of the Landau-Lifshitz type in accordance with results of [17].

**Acknowledgments.** The second author (V.S.) is grateful to USTC (Hefei, China) for hospitality. The research was partially supported by: RFBR grants 05-01-00775, 05-01-00189 and NSh grants 1716.2003.1, 2044.2003.2.

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